Linear algebra -Midterm 2

1. Let $\mathcal{P}_2$ be the space of polynomials of degree at most 2, and define the linear transformation

$$T : \mathcal{P}_2 \to \mathbb{R}^2$$
$$T(p(x)) = \begin{bmatrix} p(0) \\ p(1) \end{bmatrix}$$

For example $T(x^2+1) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

(a) Using the basis $\{1, x, x^2\}$ for $\mathcal{P}_2$, and the standard basis for $\mathbb{R}^2$, find the matrix representation of T.

(b) Find a basis for the kernel of T, writing your answer as polynomials.

Solution.

(a) The matrix representation is $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$, since $T(1) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, T(x) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, T(x^2) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

(b) The nullspace of A is spanned by $\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$, which corresponds to the polynomial $x - x^2$.

2.

(a) Find the coordinate vector of the element $1 + 3x - 6x^2$ in $\mathcal{P}_2$, relative to the basis

$$B = \{1 - x^2, x - x^2, 2 - x + x^2\}.$$

(b) In the space $\mathcal{P}_3$ of polynomials of degree at most 3, are the vectors $\{1 + 2x^3, 2 + x - 3x^2, -x + 2x^2 - x^3\}$ linearly independent?

(c) Do the vectors in part (b) form a basis for $\mathcal{P}_3$?

Solution.

(a) The coordinate vector is $\begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}$. To see this, solve the system

$$x_1 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ -6 \end{bmatrix}$$

using row reduction.

(b) Yes, they are. Row reduce the matrix

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & -1 \\ 0 & -3 & 2 \\ 2 & 0 & -1 \end{bmatrix}$$

to see this. You should get a pivot in each column.

(c) No, they do not form a basis because they do not span $\mathcal{P}_3$. (You would need 4 vectors to span $\mathcal{P}_3$.)

3. Let $M_{2\times 2}$ be the vector space of 2×2 matrices with the usual operations of addition and scalar multiplication. Define the linear transformation

$$T : M_{2\times 2} \to M_{2\times 2}$$
$$T(A) = A + A^T,$$

where A^T is the transpose of A.

(a) Find the matrix representation of T relative to the basis $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$.

(b) Find the dimension of the kernel of T.

Solution.

(a) The matrix representation is $\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$.

(b) After row reducing the matrix from part (a) there is one free variable. So the dimension of the kernel is 1.

4.

(a) Compute the determinant

$$\det \begin{bmatrix} 2 & 5 & -3 & -1 \\ 3 & 0 & 1 & -3 \\ -6 & 0 & -4 & 9 \\ 4 & 10 & -4 & -1 \end{bmatrix}.$$

(b) Let A, B be $n \times n$ matrices, such that B is invertible. Is it true that

$$\det(B^{-1}AB) = \det(A)?$$

Justify your answer!

Solution.

(a) The determinant is 120.

(b) It is true, since

$$\det(B^{-1}AB) = \det(B^{-1})\det(A)\det(B) = \det(B^{-1})\det(B)\det(A) = \det(I)\det(A) = \det(A).$$